Mathematics 166 **Exam 4.C** **April 28, 2014**

Name: SOLNS Section:________

Do not grade problem:________

Read all directions carefully. Do any 4 of the 5 problems. You must indicate CLEARLY the problem not to be counted for credit. If you do all five problems and do not indicate one to be omitted, then your grade will be the sum of your scores on the first four problems. Show all work; partial credit may be given but only if correct work and reasoning are shown and explained. Correct answers without work shown may not receive full credit.

1. For each of the following series determine whether the series converges or diverges. Justify all conclusions completely.

(8) a. $\sum_{n=4}^{\infty} \frac{2}{n(\ln n)^4}$. Integral tests: terms are ↓

2pts → $\boxed{\text{Conv.}}$ By

pts Note a valid test → $\boxed{\int \text{test}}$

$$\int_4^\infty \frac{1}{x(\ln x)^4}dx = \lim_{a\to\infty}\int_4^a \frac{1}{x(\ln x)^4}dx = \lim \int_{x=4}^{x=a} u^{-4}du = \lim\left(-\frac{1}{3(\ln x)^3}\right)\Big|_4^a = \frac{1}{3(\ln 4)^3} < \infty.$$

$u = \ln x$, $du = \frac{dx}{x}$

(4 pts - Carry out test)

(9) b. $\sum_{\ell=1}^{\infty}(-1)^\ell \frac{\ell!}{10^{2\ell}}$.

2pts → $\boxed{\text{Div.}}$

2 pts → $\boxed{n^{th}\text{ term or ratio}}$

(I) Root test $\lim_{\ell\to\infty} (-1)^\ell \frac{\ell!}{10^{2\ell}} = \lim (-1)^\ell \frac{1\cdot 2\cdot 3\cdot 4 \cdots \ell}{100\cdot 100 \cdots 100}$ when $\ell > 100$, these ↑ in abs value to ∞

∴ Div by oscillation ($\pm\infty$)

(II) Ratio test on abs. series $\lim_{\ell\to\infty} \frac{\left|\frac{(\ell+1)!}{10^{2(\ell+1)}}\right|}{\left|\frac{\ell!}{10^{2\ell}}\right|} = \lim_{\ell\to\infty} \frac{\ell+1}{100} = \infty$. So abs series div. by ratio test.

(5 pts for valid use of test)

Div. by ratio test → terms ↛ 0 ∴ given series conv. too

(8) c. $\sum_{j=1}^{\infty} \frac{2j^3+2}{5j^4+j^3}$.

2pts → $\boxed{\text{Conv}}$

2pts → $\boxed{\text{Comp test}}$

Ok to note that for j large

$\frac{2j^3+2}{5j^4+j^3} \approx \frac{2}{5j}$. Since $\sum \frac{2}{5j}$ div. (Har. series or p=1 series)

4 points for justification using test

We know by comparison that given series div

OR

Use a formal limit comparison process with $\sum \frac{1}{j}$

(NOTE: If say "$\frac{2}{5j}$ (or $\frac{1}{j}$) diverge" → −2pts)

10 pts 2. a. Create a geometric series with nonzero common ratio that converges to 166.

4 pts - Frame prob

We need to find a and r, $-1<r<1$, $a \neq 0$ with

$$166 = a + ar + ar^2 + \cdots + ar^k + \cdots = \frac{a}{1-r}$$

4 pts get a, r

Pick any $a>0$ ... say $a=2$

Then $166 = \frac{2}{1-r} \Rightarrow 1-r = \frac{1}{83} \Rightarrow r = 1 - \frac{1}{83} = \frac{82}{83}$. Series

2 pts → must write down a working g.s. for final answer

$$2\sum_{k=0}^{\infty}\left(\frac{82}{83}\right)^k = 2 + 2\left(\frac{82}{83}\right) + 2\left(\frac{82}{83}\right)^2 + \cdots$$

b. Find the Taylor Series centered about $c = 4$ for the function $f(x) = \dfrac{3}{5+x}$, and write down a clear formula for the coefficient a_k of $(x-4)^k$ for this series.

15 pts

(I.) We do this by substitution in the geometric series formula

pts $\xrightarrow[\text{n write this}]{\text{declare}}$ $\circledast$ $\quad \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$ ← just have to find "t"

10 pts, deduct for errors, bad alg, Deduct ... for missing g.s. Template)

$$\frac{3}{5+x} = \frac{3}{5+(x-4)+4} = \frac{3}{9+(x-4)} = \frac{3}{9}\cdot\left(\frac{1}{1+\frac{x-4}{9}}\right) = \frac{1}{3}\left(\underbrace{\frac{1}{1-\left(-\frac{x-4}{9}\right)}}_{\text{this is } \circledast,\ t = -\frac{x-4}{9}}\right)$$

want $x-4$ pwrs

getting 1 here com $\circledast$

$$= \frac{1}{3}\sum_{k=0}^{\infty}\left(-\frac{x-4}{9}\right)^k = \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{9^k}(x-4)^k$$

This display gives a clean description of a_k

If no clear a_k, $\boxed{-2}$

If they just write out

$$\frac{1}{3}\left(1 + \left(-\frac{x-4}{9}\right) + \left(-\frac{x-4}{9}\right)^2 + \cdots\right)$$

then need to give a_k formula

$$a_k = \frac{(-1)^k}{3\cdot 9^k}$$

(II) Can also do this by using the Taylor Series Formula:

5 pts
(-2 if 4 rep by x)

$$\longrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(4)}{k!}(x-4)^k$$

$f(x) = 3(5+x)^{-1} \Rightarrow \frac{f^{(0)}(4)}{0!} = \frac{f(4)}{1} = \frac{3}{9} \rightarrow a_0$

$f'(x) = 3(-1)(5+x)^{-2} \Rightarrow \frac{f^{(1)}(4)}{1!} = \frac{-3}{9^2} \rightarrow a_1$

$f''(x) = 3(-1)(-2)(5+x)^{-3} \Rightarrow \frac{f''(4)}{2!} = \frac{3(-1)^2 \cdot 2!\,(9)^{-3}}{2!} = -\frac{3}{9^3} \rightarrow a_2$

$f'''(x) = 3(-1)(-2)(-3)(5+x)^{-4} \Rightarrow \frac{f'''(4)}{3!} = \frac{3(-1)^3 \cdot 3!\,(9)^{-4}}{3!} = -\frac{3}{9^4} \rightarrow a_3$

$\vdots \qquad\qquad \vdots$

Can produce k^{th} der. or guess based on pattern that $a_k = \frac{(-1)^k 3}{9^k}$

$\therefore$ T.S. is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \cdot 3}{9^k}(x-4)^k$

Again, OK to give partial sum with $+\cdots$ but must have explicit formula for a indication that $a_k = \frac{(-1)^k 3}{9^k}$

3. Find the interval of convergence for the power series $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{5^k \cdot k}(x-5)^k$. Show all work and justify all conclusions.

1^{st} Use Ratio test on $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{5^k \cdot k}(x-5)^k \right|$ ← −2 if no ack. of need for abs. values

15pts For correct use of ratio test

$$\lim_{k\to\infty} \frac{(k+1)^{st}\text{ term}}{k^{th}\text{ term}} = \lim_{k\to\infty} \frac{\left|\frac{(-1)^{k+2}}{5^{k+1}(k+1)}(x-5)^{k+1}\right|}{\left|\frac{(-1)^k}{5^k \cdot k}(x-5)^k\right|}$$

want

$$= \lim_{k\to\infty} \frac{5^k}{5^{k+1}} \cdot \frac{k}{k+1} \cdot \frac{|x-5|^{k+1}}{|x-5|^k} = \frac{1}{5}\cdot 1 \cdot |x-5| < 1$$

To make this <1, need $|x-5|<5$

$\therefore$ Rad of conv $=5$.

−2 if unable to get correct interval

$$|x-5|<5 \Leftrightarrow -5<x-5<5 \Rightarrow \underbrace{0<x<10}$$

Interval of convergence but still need to examine endpoints $x=0$, $x=10$ (no info case for ratio test)

correct sub 2

4pts (i) Put $x=0$ into original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(0-5)^k}{5^k \cdot k} = -\sum_{k=1}^{\infty} \frac{1}{k}$ → Div; harmonic series

correct alg, concl. 2

4pts (ii) Put $x=10$ into original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(10-5)^k}{5^k \cdot k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ → Conv − Alt. Series test

$$\boxed{\text{Interval of conv: } 0<x\le 10}$$

NOTE (memory) $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$ ← 4pts

so $2\cos(3x) = 2\Big[\underbrace{1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!}} - \frac{(3x)^6}{6!} + \cdots\Big]$ ← 5pts

To get P_5, take terms of degree ≤ 5

$P_5 = 2 - (3x)^2 + \frac{(3x)^4}{12}$ ← 3pts

12pts

4. a. Find the Taylor polynomial of degree 5 centered at 0, that is $P_5(x)$, for the function $2\cos(3x)$.

4pts Id. G. Formula ⟶ $P_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!} x^k$

5pts (Der)

$$\left.\begin{array}{l} f(x) = 2\cos(3x);\ f(0) = 2\cos 0 = 2 \\ f'(x) = -6\sin(3x);\ f'(0) = -6\sin 0 = 0 \\ f''(x) = -18\cos(3x);\ f''(0) = -18\cos 0 = -18 \\ f'''(x) = 54\sin(3x)\ \ f'''(0) = 0 \\ f^{(iv)}(x) = 162\cos(3x);\ f^{(iv)}(0) = 162 \\ f^{(v)}(x) = -3\cdot 162\sin(3x),\ \ f^{(v)}(0) = 0 \end{array}\right)$$

$= 2 + \frac{0\cdot x}{1!} - \frac{18\cdot x^2}{2!} + \frac{0\cdot x^3}{3!} + \frac{162\cdot x^4}{4!} + \frac{0\cdot x^5}{5!}$

$\boxed{= 2 - 9x^2 + + \frac{27}{4}x^4}$ $\boxed{+3pts}$ put together

13pts b. Suppose that we use $P_5(x)$ to approximate $2\cos(3x)$ on the interval $-\frac{1}{10} \leq x \leq \frac{1}{10}$. Give a bound on the error for this approximation. Your final answer should be a number. Justify your answer. You may want to use the general error relationship

$$|f(x) - P_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}.$$

We have $n = 5$, $a = 0$ (From part a, centered at 0)

$$\left| 2\cos(3x) - P_5(x) \right| \leq \frac{M_{5+1}}{(5+1)!}|x-0|^{5+1} = \frac{M_6}{6!}|x|^6$$

Express w/ correct n, a $\boxed{3pts}$

$M_6 = \max \left| \frac{d^6}{dx^6}(2\cos(3x)) \right| = \left|2\cdot 3^6 \cos(3x)\right|$ on $-\frac{1}{10} \leq x \leq \frac{1}{10}$

max of $|\cos(3x)|$ is 1, so take

$M_6 = 2\cdot 3^6$

$M_6 \geq 2\cdot 3^6$ Justified. $\boxed{6pts}$

$$\therefore \left|2\cos(3x) - P_5(x)\right| \leq \frac{2\cdot 3^6}{6!}|x|^6 \leq \frac{2\cdot 3^6}{6!}\left(\frac{1}{10}\right)^6 = \frac{81}{40\,000\,000} \approx 2.025\times 10^{-6}$$

$|x| \leq \frac{1}{10}$ for $-\frac{1}{10} \leq x \leq \frac{1}{10}$

4pts; $|x| \leq \frac{1}{10}$ & put info together right.

11b Second Solution. The McLauren Series for $2\cos(3x)$ is

$$2\cos(3x) = \sum_{k=0}^{\infty} (-1)^k \frac{2\cdot 3^{2k} x^{2k}}{(2k)!}$$

$$= \underbrace{2 - 9x^2 + \frac{27}{4}x^4}_{P_5} - \frac{2\cdot 3^6 \cdot x^6}{6!} + -$$

Because all terms have even degree, all expressions $\dfrac{2\cdot 3^{2k} x^{2k}}{(2k)!}$ are positive,

so the McLauren series is alternating. Also, for $-\frac{1}{10} \le x \le \frac{1}{10}$

$2 > 9x^2 > \frac{27}{4}x^4 > \ldots$ the terms are $\downarrow 0$.

Use the error estimate from the Alternating Series Test:

$$\underbrace{\left| 2\cos(3x) - P_5 \right|}_{|\text{Error}|} \le \left| 1^{\text{st}} \text{ omitted term from the series} \right| = \left| \frac{2\cdot 3^6 x^6}{6!} \right|$$

$$\le \frac{2\cdot 3^6 \left(\frac{1}{10}\right)^6}{6!}$$

5. Let $\sum_{k=0}^{\infty} c_k(x-2)^k$ be a power series. Assume that using the ratio test it is shown that the series has radius of convergence 6. (All parts of this problem deal with this series.)

7pts

a. Find, with justification, the value of $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$.

R.of C. = 6 ⇒ in ratio test, need $|x-2|<6$ to make ratio result <1

$$\therefore \lim_{k\to\infty}\left|\frac{C_{k+1}(x-c)^{k+1}}{C_k(x-c)^k}\right| = \lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right||x-2| = 1 \Rightarrow \lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right|\cdot 6 = 1$$

↑ this = 1 (No info) if $|x-2|=6$

$$\boxed{\Rightarrow \lim\left|\frac{C_{k+1}}{C_k}\right| = \frac{1}{6}}$$

9pts

b. If we twice differentiate the given power series we get the power series $\sum_{k=2}^{\infty} k(k-1)c_k(x-2)^{k-2}$. Find, with justification, the radius of convergence of this new series.

2pts → Use Ratio test

3pts (use (a))

need <1 for conv

$$\underbrace{\lim_{k\to\infty}\left|\frac{(k+1)k\,C_{k+1}(x-2)^{k-1}}{k(k-1)C_k(x-2)^{k-1}}\right|}_{2pts} = \lim_{k\to\infty} \frac{k+1}{k-1}\cdot\left|\frac{C_{k+1}}{C_k}\right||x-2| = 1\cdot\frac{1}{6}|x-2| < 1$$

$$\boxed{\therefore |x-2|<6 \quad R.of C. = 6}$$ 2pts

9pts

c. Find, with justification, the radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{c_k 3^k}{k!}(x-2)^k$

2pts ← Use ratio test

$$\underbrace{\lim_{k\to\infty}\frac{\left|\frac{C_{k+1}3^{k+1}}{(k+1)!}(x-2)^{k+1}\right|}{\left|\frac{C_k 3^k}{k!}(x-2)^k\right|}}_{2pts} = \lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right|\frac{k!}{(k+1)!}\cdot\frac{3^{k+1}}{3^k}\cdot\frac{|x-2|^{k+1}}{|x-2|^k}$$

$$= \lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right|\cdot\frac{1}{k+1}\cdot 3\cdot|x-2|$$

3pts (use a)

$$= \frac{1}{6}\cdot 0\cdot 3\cdot|x-2| = 0 < 1$$

So this is <1 for all x. $\boxed{\text{So R.of C} = \infty}$ (2pts (Ans)